Stretched semigroup rings and 2-minor generation in defining ideals

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Let $H = \langle n_1, \ldots, n_e \rangle = \{\lambda_1 n_1 + \lambda_2 n_2 + \cdots + \lambda_e n_e \mid \lambda_1, \lambda_2, \ldots, \lambda_e \in \mathbb{N}\}$ be an e-generated numerical semigroup, which is a submonoid of $\mathbb{N} = \{0, 1, 2, \ldots\}$ with $\sharp(\mathbb{N} \setminus H) < \infty$. Consider the numerical semigroup ring $R = k[H]$ of H which is realized as a subring of the polynomial ring $k[t]$, where k is a field. Let $S = k[X_1, \ldots, X_e]$ be the polynomial ring with e indeterminates, and $\varphi : S \to R$ the homomorphism defined by $X_i \mapsto t^{n_i}$. We call $I_H = \operatorname{Ker} \varphi$ the defining ideal of R. Understanding the generators of the defining ideal I_H is closely related to the ring-theoretic properties of $k[H]$.

To explain the background of this talk, we need some notation on numerical semigroups. For a numerical semigroup H, an integer α is called a pseudo-Frobenius number of H, if $\alpha \notin H$ and $\alpha + h \in H$ for every $h \in H \setminus \{0\}$. $\mathrm{PF}(H)$ denotes the set of all pseudo-Frobenius numbers of H. It is well-known that the graded canonical module K_R of R is generated by $\{t^{-\alpha}\}_{\alpha \in \mathrm{PF}(H)}$ as an R-module (see [2]). Hence the Cohen-Macaulay type of R_M is equal to $\sharp \, \mathrm{PF}(H)$, where M is the graded maximal ideal of R.

The following conjecture posed by Cuong-Kien-Matsuoka-Truong (see [3, 4]) suggests that $\mathrm{PF}(H)$ controls the generation of the defining ideal I_H of $R = k[H]$.

Conjecture 1. *The following conditions are equivalent.*

(1) *I_H is generated by the 2minors of a homogeneous $2 \times e$ matrix over S.*

(2) *After a suitable permutation of generators of H,*

$$I_H = \mathrm{I}_2 \begin{pmatrix} X_1^{\ell_1} & X_2^{\ell_2} & \cdots & X_e^{\ell_e} \\ X_2^{m_2} & X_3^{m_3} & \cdots & X_1^{m_1} \end{pmatrix}.$$

(3) $\mathrm{PF}(H)$ *forms an arithmetic sequence of length $e - 1$, that is* $\mathrm{PF}(H) = \{h + \alpha, h + 2\alpha, \ldots, h + (e-1)\alpha\}$ *for some $h \geq 0$ and $\alpha > 0$.*

In this talk, we will provide a partial answer to this conjecture. To state our main result, we need one more notation. Let H be a numerical semigroup and H_0 be a subsemigroup of H. We do not assume that H_0 is numerical; in other words, we do not require $\sharp(\mathbb{N} \setminus H_0) < \infty$. We put

$$\mathrm{Ap}(H, H_0) = \{h \in H \mid h - h' \notin H \text{ for all } h' \in H_0\}$$

and we call it the Apéry set of H with respect to H_0. Then the following is the main theorem of this talk.

Theorem 2 (Main Theorem)**.** *Let $\delta, \gamma \in \{n_1, n_2, \ldots, n_e\}$ and put $H_0 = \langle \delta, \gamma \rangle$. Suppose that*

$$(*) \qquad \mathrm{Ap}(H, H_0) \setminus \{0\} = \{n_1, n_2, \ldots, n_e\} \setminus \{\delta, \gamma\}.$$

Then the following are equivalent.

(1) *After a suitable permutation of generators of H,*

$$I_H = \mathrm{I}_2 \begin{pmatrix} X_1^{\ell_1} & X_2^{\ell_2} & X_3 & \cdots & X_{e-1} & X_e \\ X_2^{m_2} & X_3 & X_4 & \cdots & X_e & X_1^{m_1} \end{pmatrix}$$

for some $\ell_1, \ell_2, m_1, m_2 \geq 1$.

(2) $\mathrm{PF}(H) = \{h + \alpha, h + 2\alpha, \ldots, h + (e-1)\alpha\}$ *for some $h \geq 0$ and $\alpha > 0$.*

Another important notion in this talk is that of stretched numerical semigroups. We begin with the classical definition of stretched local rings posed by J. Sally [5].

Definition 3 ([5])**.** Let $(A, \mathfrak{m})$ be an Artinian local ring. We say that A is stretched, if $\mathfrak{m}^2 = (0)$ or $\mathfrak{m}^2$ is a principal ideal.

Definition 4 ([1])**.** A numerical semigroup H is stretched, if there exists $0 \neq f \in k[[H]]$ such that $k[[H]]/(f)$ is stretched.

Notice that $k[[H]] = k[[t^{n_1}, t^{n_2}, \ldots, t^{n_e}]]$ is realized as a subring of the formal power series ring $k[[t]]$. Hence $k[[H]]$ is a local ring. This definition is motivated by the expectation that stretchedness of H should correspond to stretchedness of $k[[H]]$.

Theorem 5. *If H is stretched, then H satisfies the condition $(*)$ in Theorem 2. When $\mathrm{PF}(H)$ forms an arithmetic sequence of length $e-1$, we have*

$$I_H = \mathrm{I}_2 \begin{pmatrix} X_1^{\ell_1} & X_2^{\ell_2} & X_3 & \cdots & X_{e-1} & X_e \\ X_2 & X_3 & X_4 & \cdots & X_e & X_1^{m_1} \end{pmatrix}$$

for some $\ell_1, \ell_2, m_1 \geq 1$, after a suitable permutation of generators of H.

In this talk, we will

- summarize the known results around Conjecture 1 and clarify how our result advances the current progress,
- present further classes of numerical semigroups satisfying $(*)$, and
- investigate, under the condition $(*)$, which elements $f \in k[[H]]$ make $k[[H]]/(f)$ stretched.

References

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